Homotopy Theory

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WHY STUDY HOMOTOPY

- Homotopy theory is a way of classifying topological spaces.
- Homotopy is a geometrically intuitive relation

INTUITIVE HOMOTOPY

The definition of homotopy formalizes the intuitive notion of continuous deformation from one object to another. For example, one can continuously deform (without scissors) a circle into an ellipse or a square, or even shrink it into a single point.

DEFINITION

Two maps, $f, g : S \to T$ are homotopic if there is a continuous function $F : S \times [0,1] \to T$ such that $F(s,0) = f(s)$ for all $s \in S$ and $F(s,1) = g(s)$ for all s in S. In this case, F is a homotopy between f and g, and we write $f \simeq g$

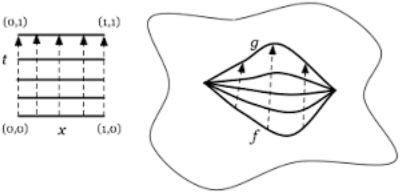


Figure: Homotopic maps

EXAMPLE

Let $f, g : R \to R$ be any two continuous functions. Define $F : R \times [0,1] \to R$ by $F(x,t) = (1-t)f(x) + tg(x)$. Then F is continuous, being a composite of two continuous functions, $F(x,0) = f(x)$ and $F(x,1) = g(x)$, so F is a homotopy between f and g. In other words, any two continuous functions on R are homotopic

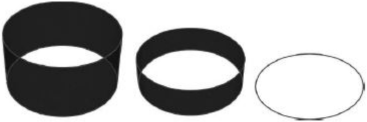
Homotopy is an equivalence relation on the set of continuous mappings S to T.

DEFINITION

Two topological spaces S, T, are homotopy equivalent if there are continuous maps $f : S \to T$ and $g : T \to S$ such that $g \circ f$ is homotopic to the identity on S and $f \circ g$ is homotopic to the identity on T. If S and T are homotopy equivalent, then we write $S \simeq$T

EXAMPLE

The inclusion $in : \{(x, y, 0) : x^2 + y^2 = 1\} \rightarrow \{(x, y, z) : x^2 + y^2 = 1, 0 \leq$ establishes a homotopy equivalence between the circle $S^1 = \{(x, y, 0) : x^2 + y^2 = 1\}$ and the cylinder $C = \{(x, y, z) : x^2 + y^2 = 1, 0 \leq$



INTRODUCTION

The fundamental group of an arcwise-connected set X is the group formed by the sets of equivalence classes of the set of all loops, i.e., paths with initial and final points at a given basepoint p, under the equivalence relation of homotopy.

DEFINITION

Let X be a topological space, and let x_0 be a point of X. We are interested in the following set of continuous functions called loops with base point x_0.

$\{f : [0,1] \to X : f(0) = x_0 = f(1)\}$

Now the fundamental group of X with base point x_0 is this set modulo homotopy h

$\{f : [0,1] \to X : f(0) = x_0 = f(1)\}/h$

equipped with the group multiplication defined by

$$(f * g)(t) = \begin{cases} f(2t) & 0 \leq t \leq \frac{1}{2} \\ g(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

Examples

space (S)	symbol	$\pi_1(S)$	$H_1(S)$
circle	$\mathbb{S}^1$	$\mathbb{Z}$	$\mathbb{Z}$
complex projective space	$\mathbb{C}P^n$	0	0
figure eight		$\mathbb{Z} * \mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}$
Klein bottle		$\frac{\mathbb{Z} \amalg \mathbb{Z}}{\langle a\, b\, a^{-1}\, b \rangle}$	$\mathbb{Z} \times \mathbb{Z}_2$
n-torus	$\mathbb{T}^n$	$\mathbb{Z}^n$	$\mathbb{Z}^n$
real projective plane	$\mathbb{R}P^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
sphere	$\mathbb{S}^2$	0	0

THE LINK

The link: The Hurewicz Theorem

For any space X and positive integer k, there exist a group homomorphism called the Hurewicz homomorphism from the $k-th$ homotopy group to the $k-th$ homology group.

Thank You!