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in tribute to Jim Stasheff and Dennis Sullivan

A TRIBUTE TO JIM STASHEFF AND DENNIS SULLIVAN

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Abstract. We provide a very brief overview of the fundamental contributions of Jim Stasheff and Dennis Sullivan to rational homotopy theory and to the theory of homotopy-coherent algebraic structures.

As a homotopy theorist with roots in rational homotopy theory, it is an honor and a challenge to write a tribute to two of the giants of 20th century algebraic topology, whose work has profoundly influenced the development of homotopy theory in general and rational homotopy theory in particular. Since even a 50-page summary of their many contributions would not do justice to their achievements, I will limit myself to the briefest of overviews of their common interest in rational homotopy theory and in homotopy-coherent algebraic structures.

At the beginning of the 1970's, Dennis was led to work on rational homotopy theory by his desire to describe the automorphism group of a finite complex as an arithmetic group, in particular over the rationals. Motivated by his intuition that there should be a way to express rational homotopy theory in terms of differential forms and thus capture geometric information, he developed his own approach to modelling rational homotopy types as commutative differential graded algebras [2], complementary to Quillen's differential graded Lie algebra models and lending themselves remarkably well to computation. The rational formality of Kähler manifolds that Dennis established together with Deligne, Griffiths, and Morgan provides a beautiful

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illustration of his philosophy that the manner in which a trivial cohomology class becomes trivial contains geometric information [1]. Dennis's proof with Vigué-Poirrier of the closed geodesic conjecture for manifolds with non-monogenic rational cohomology algebra further confirmed his intuition that rational homotopy theory could indeed capture geometric information [4].

Jim's contributions to rational homotopy theory began with his collaboration with Halperin, in which they established an elegant obstruction theory to realizing rational cohomology isomorphisms as rational homotopy equivalences [3]. He continued in this vein in his joint work with Schlessinger [5], which was available since the late 1970's only as a preprint and finally appeared on the arXiv in 2012. They showed in particular that for any simply connected commutative graded algebra H of finite type, the set of homotopy types of pairs (X, i), where X is a simply connected space and $i : H^*(X;\mathbb{Q}) \xrightarrow{\cong} H$ is a isomorphism of commutative graded algebras, has the structure of a quotient of a conical, rational algebraic variety by the action of a pro-unipotent algebraic group and is in bijection with the homotopy classes of morphisms of differential graded algebras from the Chevalley-Eilenberg construction on a certain differential graded Lie algebra L to $\mathbb{Q}$. In that sense the Lie algebra L classifies or "controls" such (X, i).

Before working on rational homotopy theory, Jim founded essentially single-handedly the study of homotopy-coherent algebraic structures. In his 1961 thesis (published in 1963 as [6]) he introduced the notion of an A_∞-structure on a topological space X, i.e., a multiplication on X together with a coherent family of higher associativity homotopies. A connected space has the homotopy type of a based loop space if and only if it admits an A_∞-structure. Moreover, transfer of strict topological monoid structures through homotopy equivalences produces A_∞-structures.

Both Jim and Dennis are firm believers in the importance of thinking of families of higher coherences synthetically, as extra structures with which a topological space (or chain complex) can be equipped, in particular because it can be difficult to recognize two instances of the same sort of higher coherence from an description in terms of formulas. More importantly, they insist on this perspective since it is evident that such structures are preserved by transfer via homotopy equivalence, i.e., the existence of such a homotopy-coherent algebraic structure is a homotopy invariant, which is not the case for strict algebraic structures.

The study of L_∞-structures, i.e., Lie algebra structures in which the Jacobi identity holds only up to an infinite coherent family of higher homotopies, was a natural offshoot of the work Jim's and Dennis's early work. Viewing deformation of algebraic structure as infinitesimal change of coordinates, it is natural to encode the structure of the object controlling these deformations as a L_∞-algebra, generalizing [5]. As Jim explored in his joint work with Sati and Schreiber [7], connections on n-bundles (particular higher categorical analogues of principal bundles with connection with relevance in string theory) naturally take their values in L_∞-algebras as well. Furthermore, Dennis showed in [8] that L_∞-structures could be profitably exploited to construct combinatorial models of continuum fluid equations, by transfering the differential graded Lie algebra structure on the sheaf of polyvector fields on an oriented manifold to an L_∞-structure on the cell complex of a cubical decomposition of the manifold. When the manifold is periodic and 3-dimensional, Dennis capitalized on Poincaré duality in order to define a combinatorial Hodge star operator and a combinatorial curl on the cell complex of a cubical decomposition.

Decades after their first important contributions to algebraic topology, Jim and Dennis are still active participants in the homotopy theory community, sharing their deep insight and proposing exciting new research directions. It is a privilege to benefit still from their wise counsel and breadth of vision.

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